Invariant-Manifold Jets for Explicit Facial Reduction in SOS Lyapunov Programs

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Code and data: https://github.com/Birdycheat/invariant-manifold-SOS-FR

***Abstract*— For Lyapunov decrease sum-of-squares (SOS) programs, sufficiently flat reduced dynamics on an invariant manifold can force the feasible Gram matrices onto a proper face of the positive semidefinite cone. We construct explicit facial reduction certificates from this dynamics before solving the program. If the reduced field is $O(\|\mathbf{x}\|^m)$, $m \geq 2$, each decrease block whose localizer is positive at the equilibrium annihilates its restricted basis-jet coefficients through degree $\lfloor m/2 \rfloor$. Rational moments of successive homogeneous layers turn these common kernels into exposing certificates orthogonal to the entire current affine coefficient-matching set. Jet-adapted coordinates give direct monomial deletion; polynomial enclosures with sufficient contact preserve the reduction. On a scalar benchmark, reduced central-point optimization and rational reconstruction certify all 80 instances. The floating-point feasibility ablation returns invalid candidates both with and without reduction.**

I. INTRODUCTION

At a non-hyperbolic equilibrium, linearization can be inconclusive. The decay is governed by the reduced dynamics on a center manifold or, more generally, on an invariant manifold [1]. For polynomial systems, SOS programming turns the search for a polynomial Lyapunov function into a semidefinite program (SDP) [2], [3].

High-order vanishing of the reduced field imposes exact algebraic cancellations in feasible SOS certificates with positive localizers. Floating-point solutions may violate these cancellations near the equilibrium. We compute the corresponding Gram constraints before optimization and use the reduced representation for exact reconstruction. Nonzero restricted jets force a proper face of the positive semidefinite (PSD) cone (Theorem 1); such degeneracy can impede interior-point methods [4], [5].

Facial reduction (FR) restricts Gram matrices to a face containing all feasible points [6]. The partial-FR variants used here search for exposing certificates in diagonal ('d') or diagonally dominant ('dd') cones [7], [8].

From the invariant jet, we prove two results:

1) *Common kernels* from a finite restricted jet, in every decrease block and any manifold dimension (Theorem 1).
2) *Explicit exposing certificates* from rational moments of each homogeneous layer, orthogonal to the entire current affine set and computed before solving (Theorem 2).

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The second result yields direct monomial deletion in jet-adapted coordinates (Corollary 2). Polynomial enclosures preserve this construction under a contact-order condition (Proposition 1, Corollary 3).

Positioning. Jones and Peet [9] prove the existence of partially quadratic Lyapunov functions, also for a vector-valued center variable. Our object is a Gram face common to a prescribed candidate family. Permenter and Parrilo [7], [8] already exploit implicit SOS constraints before SDP conversion. Here, the invariant jet constructs particular, possibly mixed directions and their homogeneous exposers without an auxiliary search. Laplagne [10] obtains kernels from zeros of a fixed polynomial; here the derivative is still unknown.

Numerical kernel recovery also addresses degenerate SOS programs. Monniaux and Corbineau [11] combine approximate semidefinite solutions, lattice reduction and exact linear algebra to recover rational kernel relations, reduce the search and verify rational witnesses. Their procedure assumes rational feasibility and infers these relations from numerical output. Kolmogorov, Naldi and Zapata [12] construct an auxiliary polynomial system with an isolated real component encoding a feasible solution, from an approximation sufficiently close to a maximum-rank feasible matrix and with suitable accuracy and rank-selection conditions. Rational feasible points are not required. Our specialized construction supplies common face constraints analytically from an invariant jet before solving the Lyapunov SDP. It complements these general recovery and certification methods: the jet face need not be minimal, and its construction does not guarantee strict feasibility. The experiments below verify the exact certificates obtained by our chain; they do not rank its total cost or success rate against these alternative exact-certification procedures.

Notation. $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ and $\mathbf{y} \in \mathbb{R}^{n_\mathbf{y}}$ are the central and transverse states. For $\alpha \in \mathbb{N}^{n_\mathbf{x}}$, $|\alpha| = \sum_i \alpha_i$ and $\mathbf{x}^\alpha = \prod_i x_i^{\alpha_i}$. For symmetric matrices, $\mathbf{Q} \succeq 0$ ($\succ 0$) means positive semidefinite (definite), and $\langle \mathbf{Q}, \mathbf{Z} \rangle = \mathrm{tr}(\mathbf{QZ})$. We use Im, ker, $\mathbf{I}$ and $\mathbf{0}$ for range, kernel, identity and zero, and $\mathbf{e}_\mu$ for the coordinate vector of monomial μ in a given basis. $T_p\mathbf{h}(\mathbf{x}) = \sum_{|\alpha| \leq p} \partial^\alpha \mathbf{h}(\mathbf{0})\, \mathbf{x}^\alpha / \alpha!$ is the Taylor polynomial of degree p of $\mathbf{h}$ at $\mathbf{0}$.

II. PROBLEM SETTING

A. Invariant manifolds and restricted derivatives

Consider the polynomial system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}), \qquad \dot{\mathbf{y}} = \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{1}$$

with an equilibrium at the origin and a local invariant graph $\mathbf{y} = \mathbf{h}(\mathbf{x})$, $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $D\mathbf{h}(\mathbf{0}) = \mathbf{0}$. Invariance reads

$$\mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = D\mathbf{h}(\mathbf{x})\,\bar{\mathbf{f}}(\mathbf{x}), \qquad \bar{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})), \tag{2}$$

where $\dot{\mathbf{x}} = \bar{\mathbf{f}}(\mathbf{x})$ is the reduced dynamics. The typical case is a center manifold [1], but only (2) and this order are used.

Assumption 1. $\bar{\mathbf{f}}(\mathbf{x}) = O(\|\mathbf{x}\|^m)$ for an integer $m \geq 2$, and $\mathbf{h}$ is of class C^{m+2} near $\mathbf{0}$.

The reduced linear part is therefore zero: nonzero imaginary eigenvalues and nonzero nilpotent linear blocks are outside this scope.

In the center–stable splitting $\mathbf{f} = O(\|(\mathbf{x}, \mathbf{y})\|^2)$, $\mathbf{g} = \mathbf{A}_s\mathbf{y} + \mathbf{g}_{\geq 2}$ with $\mathbf{A}_s$ Hurwitz, successive homogeneous parts of (2) give linear solves for $T_p\mathbf{h}$ [1].

Let $\mathcal{W}$ be an affine family of polynomial candidates $W(\mathbf{x}, \mathbf{y})$, normalized by $W(\mathbf{0}) = 0$ and $\nabla W(\mathbf{0}) = \mathbf{0}$. Let $\dot{W} = \frac{\partial W}{\partial \mathbf{x}}\mathbf{f} + \frac{\partial W}{\partial \mathbf{y}}\mathbf{g}$ be their derivative along (1) and $U(\mathbf{x}) = W(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ their restrictions.

Lemma 1 (Restricted derivative)**.** *Under Assumption 1, every $W \in \mathcal{W}$ satisfies*

$$\dot{W}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = \nabla U(\mathbf{x})^\top \bar{\mathbf{f}}(\mathbf{x}) = O(\|\mathbf{x}\|^{m+1}). \tag{3}$$

Proof. On the graph, $\mathbf{f} = \bar{\mathbf{f}}$ and, by (2), $\mathbf{g} = D\mathbf{h}\,\bar{\mathbf{f}}$, so

$$\dot{W}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = \Big(\frac{\partial W}{\partial \mathbf{x}} + \frac{\partial W}{\partial \mathbf{y}} D\mathbf{h}\Big)\bar{\mathbf{f}} = \nabla U^\top \bar{\mathbf{f}}.$$

Normalization gives $\nabla U(\mathbf{0}) = \mathbf{0}$, hence $\nabla U(\mathbf{x}) = O(\|\mathbf{x}\|)$. □

B. SOS decrease certificates and facial reduction

We consider SOS decrease certificates of the form

$$-\dot{W} - \varepsilon\ell = \sigma_0 + \sum_{j=1}^{s} \gamma_j \sigma_j, \qquad \sigma_j = \mathbf{v}_j^\top \mathbf{Q}_j \mathbf{v}_j, \quad \mathbf{Q}_j \succeq 0, \tag{4}$$

where $\varepsilon > 0$, ℓ is a prescribed nonnegative polynomial margin, and σ_0 is the free SOS term. The polynomials γ_j, $j \geq 1$, are prescribed localizers satisfying $\gamma_j(\mathbf{0}, \mathbf{0}) > 0$, and we set $\gamma_0 = 1$. Here, $\mathbf{v}_j(\mathbf{x}, \mathbf{y})$ is a prescribed monomial vector and $\mathbf{Q}_j$ is a positive semidefinite Gram matrix of σ_j. Positivity at the origin is essential: a vanishing localizer can absorb part of the required vanishing order. For example, the strip $\mathcal{S}_r = \{(\mathbf{x}, \mathbf{y}) : \|\mathbf{x}\| \leq r\}$ is described by the single localizer $\gamma_1 = r^2 - \|\mathbf{x}\|^2$. The resulting identity certifies $-\dot{W} \geq \varepsilon\ell$ on $\mathcal{S}_r$.

Matching the coefficients of (4) gives linear equations in the coefficients of W, in ε and in the Gram matrices. Their solutions form an affine set $\mathcal{A}$. The feasible decrease set is

$$\mathcal{F}_{\mathrm{dec}} = \{(W, \varepsilon, (\mathbf{Q}_j)_j) \in \mathcal{A} : \varepsilon > 0,\ \mathbf{Q}_j \succeq 0\ \forall j\}.$$

The complete Lyapunov search additionally imposes positivity of W and its chosen normalizations (Section V). A tuple $(\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_s)$ of PSD matrices is an *exposing certificate* if

$$\sum_{j=0}^{s} \langle \mathbf{Z}_j, \mathbf{Q}_j \rangle = 0 \quad \text{at every point of } \mathcal{A}. \tag{5}$$

Each term being nonnegative on the feasible set, every feasible point then satisfies $\mathbf{Z}_j\mathbf{Q}_j = \mathbf{0}$, i.e. $\mathbf{Q}_j = \mathbf{V}_j\mathbf{R}_j\mathbf{V}_j^\top$, where the columns of $\mathbf{V}_j$ form a basis of $\ker \mathbf{Z}_j$ and $\mathbf{R}_j \succeq 0$ is smaller. Repeating the argument on the restricted program gives a facial reduction sequence [5], [6].

III. COMMON KERNELS FROM RESTRICTED JETS

Let $q = \lfloor m/2 \rfloor$ and expand the restricted bases as

$$\mathbf{v}_j(\mathbf{x}, \mathbf{h}(\mathbf{x})) = \sum_{|\alpha| \leq q} \mathbf{c}_{j,\alpha}\mathbf{x}^\alpha + O(\|\mathbf{x}\|^{q+1}). \tag{6}$$

The vectors $\mathbf{c}_{j,\alpha}$ only depend on $T_q\mathbf{h}$. Let $\mathbf{C}_{j,d} = [\mathbf{c}_{j,\alpha}]_{|\alpha|=d}$ collect those of the homogeneous layer of degree d, and $\mathbf{C}_j^{\leq q} = [\mathbf{C}_{j,0}\ \mathbf{C}_{j,1}\ \cdots\ \mathbf{C}_{j,q}]$.

Example 1 (Two centers)**.** Let $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$ and $y \in \mathbb{R}$. Consider the polynomial system

$$\begin{aligned} \dot{x}_i &= -x_i^5 + x_i\big(y - 2\|\mathbf{x}\|^2\big), \qquad i = 1, 2, \\ \dot{y} &= -4(x_1^6 + x_2^6) + \big(4\|\mathbf{x}\|^2 - 1\big)\big(y - 2\|\mathbf{x}\|^2\big). \end{aligned} \tag{7}$$

Its Jacobian at the origin is $\mathrm{diag}(0, 0, -1)$.

Introduce the jet-adapted coordinate $z = y - 2\|\mathbf{x}\|^2$. Since $\dot{z} = \dot{y} - 4\mathbf{x}^\top\dot{\mathbf{x}}$, the system becomes

$$\dot{x}_i = -x_i^5 + x_i z, \qquad \dot{z} = -z.$$

Thus, $h(\mathbf{x}) = 2\|\mathbf{x}\|^2$ defines an exact invariant center manifold, with reduced dynamics $\bar{\mathbf{f}}(\mathbf{x}) = (-x_1^5, -x_2^5)^\top$. Consequently, $m = 5$ and $q = 2$.

To illustrate the restricted jets, consider the monomial vector

$$\mathbf{v} = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2, y)^\top.$$

Its nonzero jet coefficients through degree two are

$$\begin{aligned} \mathbf{c}_{(0,0)} &= \mathbf{e}_1, & \mathbf{c}_{(1,0)} &= \mathbf{e}_{x_1}, \\ \mathbf{c}_{(0,1)} &= \mathbf{e}_{x_2}, & \mathbf{c}_{(1,1)} &= \mathbf{e}_{x_1x_2}, \\ \mathbf{c}_{(2,0)} &= \mathbf{e}_{x_1^2} + 2\mathbf{e}_y, & \mathbf{c}_{(0,2)} &= \mathbf{e}_{x_2^2} + 2\mathbf{e}_y. \end{aligned}$$

These six vectors are linearly independent. The coefficient vector of $z = y - 2x_1^2 - 2x_2^2$ is

$$\mathbf{e}_z = \mathbf{e}_y - 2\mathbf{e}_{x_1^2} - 2\mathbf{e}_{x_2^2},$$

which is orthogonal to all six jet directions. Hence, in the displayed seven-dimensional monomial space,

$$(\mathrm{Im}\,\mathbf{C}^{\leq 2})^\perp = \mathrm{span}\{\mathbf{e}_z\}.$$

The curved manifold therefore imposes mixed Gram-kernel directions in the original coordinates, whereas the adapted coordinate z makes the surviving direction explicit.

Finally, the positive definite candidate

$$W(\mathbf{x}, y) = \frac{x_1^4 + x_2^4}{4} + \frac{z^2}{2}$$

satisfies

$$-\dot{W} = \left(x_1^4 - \tfrac{z}{2}\right)^2 + \left(x_2^4 - \tfrac{z}{2}\right)^2 + \tfrac{z^2}{2}, \tag{8}$$

which is positive definite in $(\mathbf{x}, z)$.

Theorem 1 (Restricted-jet kernels)**.** *Under Assumption 1, every feasible point of* (4) *satisfies* $\ell(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|^{m+1})$ *and*

$$\mathbf{Q}_j \mathbf{C}_j^{\le q} = \mathbf{0}, \qquad j = 0, 1, \dots, s. \tag{9}$$

Proof. Since the localizers are continuous and strictly positive at the origin, they are bounded below by positive constants in a neighborhood of the origin. By Lemma 1 and the nonnegativity of every term in (4), both $\ell(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ and each $\sigma_j(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ are $O(\|\mathbf{x}\|^{m+1})$. Factor $\mathbf{Q}_j = \mathbf{L}_j^\top \mathbf{L}_j$. Then $\|\mathbf{L}_j \mathbf{v}_j(\mathbf{x}, \mathbf{h}(\mathbf{x}))\| = O(\|\mathbf{x}\|^{(m+1)/2}) = o(\|\mathbf{x}\|^q)$, since $q = \lfloor m/2 \rfloor$. Hence, by (6), the Taylor polynomial $\sum_{|\alpha| \le q} \mathbf{L}_j \mathbf{c}_{j,\alpha} \mathbf{x}^\alpha$ of $\mathbf{L}_j \mathbf{v}_j(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ through degree q is $o(\|\mathbf{x}\|^q)$, so it vanishes identically. By uniqueness of polynomial coefficients, $\mathbf{L}_j \mathbf{c}_{j,\alpha} = \mathbf{0}$ for every $|\alpha| \le q$, and therefore $\mathbf{Q}_j \mathbf{c}_{j,\alpha} = \mathbf{0}$, that is, $\mathbf{Q}_j \mathbf{C}_j^{\le q} = \mathbf{0}$. □

The forced face is proper only if $\mathbf{C}_j^{\le q} \neq \mathbf{0}$ for some block.

Corollary 1 (Admissible margins and jet-kernel dimension)**.** *Let*

$$\mathcal{K}_j = \operatorname{span}\{\mathbf{c}_{j,\alpha} : |\alpha| \le q\} = \operatorname{Im} \mathbf{C}_j^{\le q}.$$

Then: (i) If $\ell(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ *has a nonzero Taylor term of degree at most* m, (4) *is infeasible. In particular,* $\ell = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ *is inadmissible. (ii) The jet-based reduction removes* $\dim \mathcal{K}_j$ *directions from Gram block* j*. If* $\mathbf{v}_j$ *contains all pure central monomials* $\mathbf{x}^\alpha$ *with* $|\alpha| \le q$*, then*

$$\dim \mathcal{K}_j = \binom{n_{\mathbf{x}} + q}{q}.$$

Proof. (i) By Theorem 1, feasibility requires $\ell(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|^{m+1})$, excluding any nonzero Taylor term of degree at most m. (ii) The rows indexed by pure central monomials form an identity submatrix: $\mathbf{C}_j^{\le q}$ has full column rank, with $\binom{n_{\mathbf{x}}+q}{q}$ columns, one per multi-index $|\alpha| \le q$. □

Remark 1 (Role of the invariant manifold)**.** Invariance supplies the uniform derivative order, which need not hold on $\mathbf{y} = \mathbf{0}$; curvature fixes the kernel orientation, even when its dimension is fixed by Corollary 1.

IV. EXPLICIT FACIAL REDUCTION BY HOMOGENEOUS LAYERS

A. Rational moment matrices

Moment-based exposers for monomial elimination are classical [13]. For $d \ge 0$, let $\mathbf{m}_d(\mathbf{x})$ collect the $\binom{n_{\mathbf{x}}+d-1}{d}$ monomials $\mathbf{x}^\alpha$ with $|\alpha| = d$, in a fixed order, and define the moment matrix $\mathbf{H}_d$ by

$$(\mathbf{H}_d)_{\alpha,\beta} = \frac{1}{2^{n_{\mathbf{x}}}} \int_{[-1,1]^{n_{\mathbf{x}}}} \mathbf{x}^{\alpha+\beta}\, d\mathbf{x} = \prod_{i=1}^{n_{\mathbf{x}}} \frac{\eta(\alpha_i + \beta_i)}{\alpha_i + \beta_i + 1}, \tag{10}$$

where $\eta(n) = 1$ if n is even and $\eta(n) = 0$ otherwise.

Lemma 2. $\mathbf{H}_d$ *has rational entries and* $\mathbf{H}_d \succ 0$*.*

Proof. For $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a}^\top \mathbf{H}_d \mathbf{a} = 2^{-n_{\mathbf{x}}} \int_{[-1,1]^{n_{\mathbf{x}}}} (\mathbf{a}^\top \mathbf{m}_d(\mathbf{x}))^2 d\mathbf{x} > 0$, since a nonzero polynomial vanishes only on a null set. □

For $n_{\mathbf{x}} = 1$, $\mathbf{H}_d = 1/(2d+1)$. Moment matrices preserve the polynomial relations (see the argument following (12)); an arbitrary positive definite coefficient weight need not do so.

B. Layerwise exposing certificates

Theorem 1 identifies mandatory kernel directions for every feasible Gram matrix. However, facial reduction requires exposing certificates orthogonal to the entire affine coefficient-matching set, as in (5), not merely to its PSD-feasible points.

Assumption 2. Assumption 1 holds, and the prescribed margin satisfies

$$\ell(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|^{m+1}).$$

By Lemma 1, the restriction of $-\dot{W} - \varepsilon\ell$ then vanishes to order at least $m+1$ at every point of $\mathcal{A}$.

Set $\mathcal{A}_0 = \mathcal{A}$. For $d = 1, 2, \dots, q+1$, define the affine set

$$\mathcal{A}_d = \big\{(W, \varepsilon, (\mathbf{Q}_j)_j) \in \mathcal{A} : \mathbf{Q}_j \mathbf{c}_{j,\alpha} = \mathbf{0} \quad \forall j,\ |\alpha| < d\big\}.$$

Set $\mathbf{V}_{j,0} = \mathbf{I}$. For $d \ge 1$, choose the columns of $\mathbf{V}_{j,d}$ to form a basis of

$$\big(\operatorname{span}\{\mathbf{c}_{j,\alpha} : |\alpha| < d\}\big)^\perp.$$

For symmetric $\mathbf{Q}_j$, the preceding kernel constraints are equivalent to

$$\mathbf{Q}_j = \mathbf{V}_{j,d} \mathbf{R}_j \mathbf{V}_{j,d}^\top,$$

where $\mathbf{R}_j$ is symmetric. Since $\mathbf{V}_{j,d}$ has full column rank, $\mathbf{Q}_j \succeq 0$ if and only if $\mathbf{R}_j \succeq 0$.

With matching monomial orders, define

$$\mathbf{Z}_{j,d} = \gamma_j(\mathbf{0}, \mathbf{0})\, \mathbf{C}_{j,d} \mathbf{H}_d \mathbf{C}_{j,d}^\top. \tag{11}$$

Since $\mathbf{H}_d \succ 0$ by Lemma 2 and $\gamma_j(\mathbf{0}, \mathbf{0}) > 0$, we have $\mathbf{Z}_{j,d} \succeq 0$. Moreover, $\mathbf{H}_d^{1/2}$ is invertible, so

$$\begin{aligned} \operatorname{Im} \mathbf{Z}_{j,d} &= \operatorname{Im}(\mathbf{C}_{j,d} \mathbf{H}_d^{1/2} \mathbf{H}_d^{1/2} \mathbf{C}_{j,d}^\top) \\ &= \operatorname{Im}(\mathbf{C}_{j,d} \mathbf{H}_d^{1/2}) \\ &= \operatorname{Im} \mathbf{C}_{j,d}, \end{aligned}$$

where we used $\operatorname{Im}(\mathbf{B}\mathbf{B}^\top) = \operatorname{Im} \mathbf{B}$ and the positive scalar factor does not change the range.

Theorem 2 (Layerwise exposing certificates)**.** *Under Assumption 2, the matrices* $\mathbf{Z}_{j,d}$ *defined in* (11) *satisfy, for each* $d = 0, 1, \dots, q$*:*

1) Affine orthogonality:

$$\sum_{j=0}^{s} \langle \mathbf{Z}_{j,d}, \mathbf{Q}_j \rangle = 0$$

at every point of $\mathcal{A}_d$, without requiring the Gram matrices to be positive semidefinite.

2) Forced kernel constraints: *every point of $\mathcal{A}_d$ with $\mathbf{Q}_j \succeq 0$ for all j satisfies $\mathbf{Q}_j \mathbf{C}_{j,d} = \mathbf{0}$, and therefore belongs to $\mathcal{A}_{d+1}$.*
3) Facial reduction: *the compressed tuples*

$$\left(\mathbf{V}_{j,d}^\top \mathbf{Z}_{j,d} \mathbf{V}_{j,d}\right)_{j=0}^{s},$$

after skipping any zero tuple, form a facial reduction sequence for (4)*. Its final face,*

$$\mathbf{Q}_j = \mathbf{V}_j \mathbf{R}_j \mathbf{V}_j^\top, \qquad \mathbf{R}_j \succeq 0, \quad \mathbf{V}_j = \mathbf{V}_{j,q+1},$$

contains every feasible point.

Proof. Fix $d \in \{0, 1, \dots, q\}$ and a point of $\mathcal{A}_d$.

Step 1: Isolating the homogeneous layer. By Assumption 2 and Lemma 1, restricting the polynomial identity (4) to the invariant graph gives

$$\sum_{j=0}^{s} \gamma_j(\mathbf{x}, \mathbf{h}(\mathbf{x}))\sigma_j(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|^{m+1}).$$

All its homogeneous components through degree m therefore vanish. By definition of $\mathcal{A}_d$, the constraints $\mathbf{Q}_j \mathbf{c}_{j,\alpha} = \mathbf{0}$ already hold for $|\alpha| < d$. Since $\mathbf{Q}_j$ is symmetric, every term of $\sigma_j(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ involving one of these lower-order coefficients vanishes.

Consequently, the first potentially nonzero homogeneous term has degree $2d$ and is

$$\mathbf{m}_d(\mathbf{x})^\top \mathbf{C}_{j,d}^\top \mathbf{Q}_j \mathbf{C}_{j,d} \mathbf{m}_d(\mathbf{x}).$$

Since the restricted quadratic form has no terms below degree $2d$, only the constant term $\gamma_j(\mathbf{0}, \mathbf{0})$ of the localizer contributes to its degree-$2d$ component. Taking this component in the restricted identity gives

$$\sum_{j=0}^{s} \gamma_j(\mathbf{0}, \mathbf{0})\, \mathbf{m}_d(\mathbf{x})^\top \mathbf{C}_{j,d}^\top \mathbf{Q}_j \mathbf{C}_{j,d} \mathbf{m}_d(\mathbf{x}) = 0 \quad \forall \mathbf{x}. \tag{12}$$

Step 2: From polynomial identity to exposure. Integrating (12) over $[-1, 1]^{n_\mathbf{x}}$ with normalized uniform measure, and using the moment matrix $\mathbf{H}_d$ from (10) together with linearity and cyclic invariance of the trace, yields

$$\sum_{j=0}^{s} \left\langle \gamma_j(\mathbf{0}, \mathbf{0}) \mathbf{C}_{j,d} \mathbf{H}_d \mathbf{C}_{j,d}^\top, \mathbf{Q}_j \right\rangle = 0.$$

The matrices inside the inner products are exactly $\mathbf{Z}_{j,d}$, proving assertion 1.

Now suppose $\mathbf{Q}_j \succeq 0$ for every j. Since $\mathbf{Z}_{j,d} \succeq 0$, every term $\langle \mathbf{Z}_{j,d}, \mathbf{Q}_j \rangle$ is nonnegative. Their sum being zero, each term must vanish. For two positive semidefinite matrices, zero trace product implies $\mathbf{Q}_j \mathbf{Z}_{j,d} = \mathbf{0}$. The range identity proved above then gives $\mathbf{Q}_j \mathbf{C}_{j,d} = \mathbf{0}$. These are precisely the additional constraints defining $\mathcal{A}_{d+1}$, proving assertion 2.

Step 3: Successive facial reductions. On $\mathcal{A}_d$, substituting $\mathbf{Q}_j = \mathbf{V}_{j,d} \mathbf{R}_j \mathbf{V}_{j,d}^\top$ into assertion 1 gives

$$\sum_{j=0}^{s} \left\langle \mathbf{V}_{j,d}^\top \mathbf{Z}_{j,d} \mathbf{V}_{j,d}, \mathbf{R}_j \right\rangle = 0.$$

Each compressed exposing matrix is positive semidefinite. A nonzero tuple therefore exposes a proper face of the current reduced PSD cone; a zero tuple adds no kernel constraint and is skipped. By assertion 2, restricting to the exposed face removes no feasible point.

Applying this argument successively for $d = 0, 1, \dots, q$ proves assertion 3. □

Remark 2 (Exact computation)**.** For fixed monomial bases, $\mathbf{Z}_{j,d}$ depends only on $T_q \mathbf{h}$ and $\gamma_j(\mathbf{0}, \mathbf{0})$, and can be constructed before solving the SDP, without numerical rank decisions. Since $\mathbf{H}_d$ is rational, $\mathbf{Z}_{j,d}$ is rational whenever the jet coefficients and localizer values are rational.

C. Jet-adapted coordinates

Corollary 2 (Jet-adapted monomial deletion)**.** *Under Assumption 2, let $\boldsymbol{\varphi} = T_p \mathbf{h}$, $p \geq q$, with $\mathbf{h} \in C^{p+1}$, and set $\mathbf{z} = \mathbf{y} - \boldsymbol{\varphi}(\mathbf{x})$. In monomial bases $\mathbf{v}_j(\mathbf{x}, \mathbf{z})$, the jet-induced facial reduction of Theorem 2 is realized by deleting, from each basis, all pure central monomials $\mathbf{x}^\alpha$ with $|\alpha| \leq q$.*

Proof. On the invariant graph, the adapted coordinate satisfies

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) - \boldsymbol{\varphi}(\mathbf{x}) = O(\|\mathbf{x}\|^{p+1})$$

by Taylor's theorem.

Consider any monomial $\mathbf{x}^\alpha \mathbf{z}^\beta$ in $\mathbf{v}_j(\mathbf{x}, \mathbf{z})$, with $\alpha \in \mathbb{N}^{n_\mathbf{x}}$ and $\beta \in \mathbb{N}^{n_\mathbf{y}}$. If $|\beta| \geq 1$, its restriction to the invariant graph satisfies

$$\mathbf{x}^\alpha \mathbf{z}^\beta = O\big(\|\mathbf{x}\|^{|\alpha| + (p+1)|\beta|}\big) = O(\|\mathbf{x}\|^{q+1}),$$

since $p \geq q$. Thus, no monomial containing a transverse variable contributes to the restricted jet through degree q.

Conversely, pure central monomials remain unchanged on the graph. Hence, for $|\alpha| \leq q$,

$$\mathbf{c}_{j,\alpha} = \begin{cases} \mathbf{e}_{\mathbf{x}^\alpha}, & \mathbf{x}^\alpha \in \mathbf{v}_j, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Consequently,

$$\operatorname{Im} \mathbf{C}_j^{\leq q} = \operatorname{span}\{\mathbf{e}_{\mathbf{x}^\alpha} : \mathbf{x}^\alpha \in \mathbf{v}_j,\ |\alpha| \leq q\}.$$

By Theorem 2, every feasible Gram matrix satisfies $\mathbf{Q}_j \mathbf{C}_j^{\leq q} = \mathbf{0}$. Therefore, the rows and columns of $\mathbf{Q}_j$ corresponding to these pure central monomials vanish. After reordering the basis,

$$\mathbf{v}_j = \begin{pmatrix} \mathbf{v}_{j,c} \\ \mathbf{v}_{j,r} \end{pmatrix}, \qquad \mathbf{Q}_j = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{pmatrix}, \quad \mathbf{R}_j \succeq 0,$$

where $\mathbf{v}_{j,r}$ contains the remaining monomials. Thus,

$$\sigma_j = \mathbf{v}_j^\top \mathbf{Q}_j \mathbf{v}_j = \mathbf{v}_{j,r}^\top \mathbf{R}_j \mathbf{v}_{j,r}.$$

□

When the localizers depend on $\mathbf{x}$ only and $-\dot{W} - \varepsilon\ell$ has degree ν in the transverse variables, comparing the highest transverse homogeneous parts in (4) also deletes the Gram monomials of transverse degree above $\nu/2$.

Polynomial bounds and Taylor remainders already support rigorous SOS analysis of non-polynomial systems [14], [15].

Proposition 1 (Non-polynomial fields)**.** *Let the field be* $\mathbf{F} = \mathbf{F}_0 + \sum_i g_i \mathbf{F}_i$*, with polynomial* $\mathbf{F}_0, \mathbf{F}_i$ *and scalar* g_i*, the graph satisfying* (2) *and Assumption 1 for* $\mathbf{F}$*, and let* $g_i^- \le g_i \le g_i^+$ *on* $\{\gamma_j \ge 0\ \forall j\}$ *for polynomials* $g_i^\pm$*. Let* $\mathbf{F}_v$ *be the polynomial fields at the vertices of this box. (i) If* (4) *holds for every* $\mathbf{F}_v$*, with common* W *and* ε*, then* $-\dot{W} \ge \varepsilon\ell$ *along* $\mathbf{F}$ *on that set. (ii) If* $(\mathbf{F}_v - \mathbf{F})(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|^m)$ *for all* v*, Theorem 1 holds for the program collecting the vertices, with common* W*,* ε *and one Gram tuple per vertex. Under Assumption 2, Theorem 2 and Corollary 2 also hold, retaining the latter's regularity and jet-order hypotheses.*

Proof. (i) At every point of the prescribed domain, the bounds on g_i express $\mathbf{F}$ as a convex combination of the polynomial vertex fields $\mathbf{F}_v$. Since $\dot{W} = \nabla W^\top \mathbf{F}$ is linear in the vector field, the decrease inequalities at all vertices imply $-\dot{W} \ge \varepsilon\ell$ for the true field.

(ii) For every normalized candidate $W \in \mathcal{W}$, Lemma 1 and the contact condition yield, on the invariant graph,

$$\begin{aligned}\nabla W^\top \mathbf{F}_v &= \nabla W^\top \mathbf{F} + \nabla W^\top (\mathbf{F}_v - \mathbf{F}) \\ &= O(\|\mathbf{x}\|^{m+1}),\end{aligned}$$

since $\nabla W(\mathbf{x}, \mathbf{h}(\mathbf{x})) = O(\|\mathbf{x}\|)$. Thus, the restricted derivative has the required vanishing order at every vertex, without requiring the graph to be invariant under each $\mathbf{F}_v$.

The proof of Theorem 1 therefore applies to each vertex Gram tuple. Under Assumption 2, the prescribed margin has the same vanishing order, so the affine coefficient-matching argument of Theorem 2 also applies vertexwise. Corollary 2 then follows under its stated hypotheses. □

Corollary 3 (Taylor enclosures)**.** *Let* $\mathbf{F}$ *be of class* C^{m+2} *near the ball* $\mathcal{B}_r = \{\mathbf{X} = (\mathbf{x}, \mathbf{y}) : \|\mathbf{X}\| \le r\}$*, the graph satisfying* (2) *and Assumption 1 for* $\mathbf{F}$*, and let* $\{\gamma_j \ge 0\ \forall j\} \subseteq \mathcal{B}_r$*. Set* $\mathbf{P} = T_{2q+1}\mathbf{F}$ *and* $b = M(\mathbf{X}^\top \mathbf{X})^{q+1}$*, with*

$$M = \max_i \sup_{\mathcal{B}_r} \|D^{2q+2} F_i\|/(2q+2)! \tag{13}$$

or any certified M *such that* $|F_i - P_i| \le b$ *on* $\mathcal{B}_r$ *for all* i*; the derivative norm is the Euclidean multilinear operator norm. Then Proposition 1 applies with* $\mathbf{F}_0 = \mathbf{P}$*,* $\mathbf{F}_i$ *the* i*th canonical vector,* $g_i = F_i - P_i$ *and* $g_i^\pm = \pm b$*. Its vertices are the polynomial fields* $\mathbf{P} + b\,\mathbf{s}$*,* $\mathbf{s} \in \{-1, 1\}^{n_\mathbf{x} + n_\mathbf{y}}$*, and they differ from* $\mathbf{F}$ *by* $O(\|\mathbf{x}\|^{m+1})$ *on the graph. Exposure conclusions additionally require Assumption 2.*

Proof. Taylor's formula along the segment $t \mapsto t\mathbf{X}$ gives $|F_i - P_i| \le M\|\mathbf{X}\|^{2q+2} = b$ with (13), and b is a polynomial since $2q+2$ is even. As $2q+2 \ge m+1$ and $\|(\mathbf{x}, \mathbf{h}(\mathbf{x}))\| = O(\|\mathbf{x}\|)$, b and g_i are $O(\|\mathbf{x}\|^{m+1})$ on the graph. □

In the center–stable case, $\mathbf{P}$ suffices for $T_q\mathbf{h}$ and the order m, since $2q+1 \ge m$. The order-$m+1$ remainder preserves reduced terms through degree m; an order-m contact can preserve kernels while changing these terms.

Remark 3 (Diagonal approximations)**.** For a vector $\mathbf{c}$, $\mathbf{c}\mathbf{c}^\top$ is diagonally dominant if and only if $\mathbf{c}$ has at most one nonzero entry, or two of equal magnitude: summing the dominance inequalities over a support of size t gives $\sum_i |c_i| \ge (t-1)\sum_i |c_i|$. Thus $\mathbf{c}_{(2,0)} = \mathbf{e}_{x_1^2} + 2\mathbf{e}_y$ in Example 1 is not of this type in the displayed basis. Scaling y to $y/2$ makes the two entries equal. Thus this obstruction depends on the basis scaling; sums can also be dominant. The comparisons below use the stored, unscaled bases.

V. NUMERICAL STUDY

A. Programs and exact validation

We distinguish floating-point feasibility, a separate interior-margin optimization, and exact reconstruction. Success means solver status zero. A witness proves $\dot{W} > 0$ at a point in the program's domain; absence of a witness is inconclusive. Stored floating-point candidates and their coordinate maps are interpreted as exact rationals for these checks. For analytic fields, rational remainder bounds certify the derivative sign.

For exact certification, let $\mathcal{V}$ index the polynomial fields (one field for a polynomial problem, all enclosure vertices otherwise). At fixed radius and on the retained Gram spaces, we maximize t subject to the coefficient identities (4) at every $\nu \in \mathcal{V}$ and

$$\begin{aligned} W = \mathbf{u}^\top \mathbf{Q}_W \mathbf{u}, \quad \sigma_{\nu,j} = \mathbf{w}_{\nu,j}^\top \mathbf{R}_{\nu,j} \mathbf{w}_{\nu,j}, \\ \mathbf{Q}_W \succeq t\mathbf{I}, \quad \mathbf{R}_{\nu,j} \succeq t\mathbf{I}, \quad \varepsilon \ge t, \\ \operatorname{tr}\mathbf{Q}_W + \sum_{\nu \in \mathcal{V}} \sum_j \operatorname{tr}\mathbf{R}_{\nu,j} = 1. \end{aligned} \tag{14}$$

The candidate W and margin ε are common to all vertices; the decrease Gram matrices are not. Unit trace fixes the common scale. The numerical $t^\star$ depends on the bases and normalization.

Rational projection restores the coefficient identities, after which positive definiteness of every retained Gram block and $\varepsilon > 0$ are checked exactly. The stored bases vanish at the origin: $\mathbf{u}$ contains a pure power of each state, and every free-SOS basis $\mathbf{w}_{\nu,0}$ a pure power of each transverse coordinate. Thus $W > 0$ off the origin; decrease is strict through $\ell(\mathbf{x}) > 0$ for $\mathbf{x} \ne 0$ and through the free SOS for $\mathbf{x} = 0$, $\mathbf{z} \ne 0$. Compact sublevels contained in the domain's interior certify attraction.

B. Scalar benchmark: protocol

We retain 80 systems $\dot{x} = g_0(x) + x\mathbf{a}^\top \mathbf{y}$, $\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y} + x\mathbf{B}\mathbf{y} + x^2\mathbf{w}_2 + x^3\mathbf{w}_3$: 20 seeds for each $(n_\mathbf{y}, m) \in \{1, 3\} \times \{5, 7\}$. The transverse matrix is Hurwitz. After rounding the sampled data to three decimals, g_0 is computed rationally so that $\bar{f} = -x^m + O(x^{m+1})$ holds exactly. Orders $m \ge 5$ make the curved jet visible, with $q = (m-1)/2 \ge 2$.

For each instance, eight bisections on $[0, 1.5]$ give a numerical feasible endpoint r_{anc} for $W_{\text{anc}} = x^{m-1}/(m-1) + \mu\|\mathbf{y} - T_{m+1}\mathbf{h}\|^2$. We use $r = r_{\text{anc}}/2$ as the comparison radius. The anchored certificate is verified exactly after rounding μ

Algorithm 1 From an invariant jet to an exact certificate

Require: Assumptions 1–2, chosen rational program data, and a rational jet or a rational center–stable splitting.

1: Compute the required jet; check its finite invariance residual and reduced-field order. Certify enclosure bounds and vertex contact if needed.
2: Form $\mathbf{C}_{j,d}$, $\mathbf{H}_d$ and $\mathbf{Z}_{j,d}$; restrict the Gram blocks to the proved face, or delete the adapted central monomials.
3: Solve (14) on the retained spaces; record the bases, numerical margin and solver status. Refine the reduction if needed.
4: Round and project onto the exact coefficient identities, retaining the actual rational domain used in reconstruction.
5: Verify identities, matrix positivity, $\varepsilon > 0$, basis zero sets and any analytic enclosure. Return a certificate only if all checks pass.

to the 10^{-6} grid and rounding r down to the 10^{-4} grid. Except for the adaptation below, the domain is the strip $|x| \le r$, with unbounded transverse coordinates. For the comparison SOS searches except the Jones–Peet adaptation, the margin is $\ell = x^{d_\ell}$, $d_\ell = \min(2m-2, D+m-1)$; this exponent is even and at least $m+1$.

The six original-coordinate searches are free SOS with $D = 2, 4, 6$, partial FR 'd'/'dd' with $D = 4$, and a Jones–Peet adaptation [9]. Free candidates contain all monomials of degrees $2, \dots, D$, with W SOS, $\varepsilon = 10^{-3}$ and SOS multiplier degree $2\lfloor(\deg \dot{W} - 2)/2\rfloor$. The adaptation uses $W = J(x) + \mathbf{y}^\top \mathbf{H}(x) + \mathbf{y}^\top \mathbf{P}\mathbf{y}$, $\deg J \le 6$, $\deg \mathbf{H} \le 5$, $W - 10^{-5}\|(x,\mathbf{y})\|^2$ SOS, a full ball and zero decrease margin. Two further searches use $\mathbf{z} = \mathbf{y} - T_q\mathbf{h}$ and $D = 4$, without and with diagonal FR. They form the fixed-candidate-space ablation; free SOS at $D = 4$ versus partial FR at $D = 4$ also fixes the original-coordinate candidate space. Equal degree bounds across these nonlinear coordinates need not preserve the candidate space: $(y - x^2)^4$ has adapted degree four but original degree eight. MOSEK [16], YALMIP [17] and `frlib` [8] are used.

C. Scalar benchmark: results

The six original-coordinate formulations return 470 successful outputs out of 480; all 470 have exact witnesses. The two adapted formulations also return 80 outputs each. Without FR, all have witnesses; with FR, 14 have witnesses at $|x| = 10^{-9}$–10^{-8}. The other 66 are refuted by an exact transverse-defect test. For $N = \partial_{\mathbf{y}}^2 \dot{W}(0,0) \prec 0$ and the first nonzero coefficient $\boldsymbol{\delta}_i$ of $\partial_{\mathbf{y}} \dot{W}(x, \mathbf{h}(x))^\top$, $1 \le i \le q$, setting $\mathbf{y} = \mathbf{h}(x) - N^{-1}\boldsymbol{\delta}_i x^i$ gives

$$\dot{W} = -\tfrac{1}{2}\boldsymbol{\delta}_i^\top N^{-1}\boldsymbol{\delta}_i x^{2i} + O(x^{2i+1}) > 0$$

for sufficiently small nonzero x, since $2i < m+1$.

Table IB instead reports new points from (14), followed by rounding to 10^{-12}, exact projection [18] and rational positivity checks. The explicit bases omit pure central monomials of degree at most q and decrease-basis monomials of transverse degree above $D/2$; the latter deletion follows from the transverse-degree argument after Corollary 2. All 80 instances are certified, with numerical $t^\star \ge 4 \cdot 10^{-3}$. The actual rational radius is $\lfloor 10^4 r \rfloor / 10^4$ (76 decreases, four unchanged), used *before* solving the centering problem.

Both tables give $100 r_{\mathrm{bad}}/r$, where r_{bad} is the largest *recorded search radius* with a verified witness. For multiple centers, the rationalized point can have a slightly different norm; its domain membership is checked exactly.

D. Coupled and higher-dimensional benchmarks

For each $n_{\mathbf{y}} \in \{1, 2\}$ we use 20 retained dyadic instances, with $\mathbf{y} = \mathbf{z} + \boldsymbol{\varphi}(\mathbf{x})$, quadratic $\boldsymbol{\varphi}$, and triangular Hurwitz

TABLE I

SCALAR BENCHMARK: 80 EXACTLY CRITICAL SYSTEMS. A: FLOATING-POINT FEASIBILITY OUTPUTS. B: SEPARATE CENTERING AND EXACT RECONSTRUCTION. JET ROWS USE ADAPTED COORDINATES AND $D = 4$.

A. Feasibility and witness search				
Formulation	succ.	witness	none	$100 r_{\mathrm{bad}}/r$
Free SOS, $D = 2$	70/80	70	0	4.03–18.6
Free SOS, $D = 4$	80/80	80	0	2.02–10.6
Jones–Peet adaptation	80/80	80	0	4.36–13.1
Partial FR 'd', $D = 4$	80/80	80	0	$3.02 \cdot 10^{-3}$–0.836
Jet SOS, no FR	80/80	80	0	4.40–11.5
Jet SOS + FR 'd'	80/80	14	66	$\le 2.78 \cdot 10^{-6}$
B. Separate exact certification				
Reduced centering + reconstruction				80/80

All 66 "none" outputs are refuted by the transverse-defect test. Percent ranges span four group medians over witnesses; the last row bounds all 14 witness radii. Free SOS $D = 6$ and partial FR 'dd', omitted, have the same success and witness counts as $D = 4$ and 'd'.

TABLE II

MULTIDIMENSIONAL BENCHMARKS: $m = 5$, $D = 4$, $r = 1/2$. A AND B CONCERN SEPARATE SOLVES. B IS RESTRICTED TO THE INDEPENDENTLY CLASSIFIED STABLE INSTANCES.

A. Floating-point feasibility outputs				
Formulation	succ.	witness	none	$100 r_{\mathrm{bad}}/r$
Two centers: 34 stable, 6 unstable				
Free SOS, $(\mathbf{x}, \mathbf{y})$	34/40	34	0	2.47
Partial FR 'd'	34/40	34	0	$5.62 \cdot 10^{-3}$
Partial FR 'dd'	34/40	34	0	$5.62 \cdot 10^{-3}$
Jet SOS, no FR	34/40	34	0	1.78
Cor. 2, $(\mathbf{x}, \mathbf{z})$	34/40	7	27	$1.00 \cdot 10^{-3}$
Three centers: 32 stable, 8 unstable				
Free SOS, $(\mathbf{x}, \mathbf{y})$	32/40	32	0	1.78
Partial FR 'd'	32/40	32	0	$1.39 \cdot 10^{-3}$
Thm. 2, $n_{\mathbf{y}} = 1$	16/20	3	13	$5.62 \cdot 10^{-4}$
Jet SOS, no FR	32/40	32	0	1.78
Cor. 2, $(\mathbf{x}, \mathbf{z})$	32/40	10	22	$1.78 \cdot 10^{-3}$

B. Separate centering and exact certification			
Cohort	$\lvert t^\star_{\mathrm{no\,FR}} \rvert$	$t^\star_{\mathrm{face}}$	exact
Two centers, $(\mathbf{x}, \mathbf{z})$	$< 3 \cdot 10^{-10}$	$\ge 3.19 \cdot 10^{-4}$	34/34
Three centers, $(\mathbf{x}, \mathbf{z})$	$< 3 \cdot 10^{-10}$	$\ge 4.24 \cdot 10^{-3}$	32/32
Three centers, $(\mathbf{x}, \mathbf{y})$, $n_{\mathbf{y}} = 1$	$< 3 \cdot 10^{-10}$	$\ge 1.42 \cdot 10^{-3}$	–

A: median recorded search-radius ratios over the outputs with witnesses. B: numerical bounds over each stable cohort in fixed coordinates. The original-coordinate three-center program has no stored rational reconstruction. All retained adapted-coordinate certificates use the first centering iteration, with no additional heuristic deletion.

$\mathbf{\Lambda}$. Pure-square coefficients of $\boldsymbol{\varphi}$ avoid $0, \pm 1$. Here $D = 4$, $m = 5$, $r = 1/2$ and $\ell = \sum_i x_i^8$. The domain bounds $\|\mathbf{x}\|$, not the transverse coordinates. Explicit centering programs, both with and without jet reduction, remove decrease-basis monomials of transverse degree above two.

Two centers: analytic field and a noninvariant jet graph. We use

$$\begin{aligned} \dot{x}_i &= (2 - \cos(x_1 - x_2))\widehat{f}_i(\mathbf{x}) + x_i \mathbf{a}_i^\top \mathbf{z}, \\ \dot{\mathbf{z}} &= (\mathbf{\Lambda} + x_1 \mathbf{B}_1 + x_2 \mathbf{B}_2)\mathbf{z} + e^{-\|\mathbf{x}\|^2}\widehat{\mathbf{e}}(\mathbf{x}), \end{aligned} \tag{15}$$

where $\widehat{f}_i$ is dense homogeneous of degree five, with coefficients $-d_i$ on x_i^5 and mixed coefficients in $[-1, 1]$, and $\widehat{\mathbf{e}}$ has degree four. The quartic forcing makes $\mathbf{z} = 0$ noninvariant. Its graph correction satisfies $\mathbf{\Lambda}\boldsymbol{\eta}_4 + \widehat{\mathbf{e}} = 0$, hence $T_3\mathbf{h} = \boldsymbol{\varphi}$, $T_4\mathbf{h} = \boldsymbol{\varphi} - \mathbf{\Lambda}^{-1}\widehat{\mathbf{e}}$ and $g_i = \widehat{f}_i - x_i \mathbf{a}_i^\top \mathbf{\Lambda}^{-1}\widehat{\mathbf{e}}$ is the leading reduced field. Independent reference classes follow from exact Sturm isolation of angular zeros and radial-sign checks on every invariant direction, including the vertical direction; the no-ray case is checked by strict radial negativity. These tests give 34 stable and six unstable instances.

The bounds $1 \le 2 - \cos u \le 1 + u^2/2$ and $1 - \|\mathbf{x}\|^2 \le e^{-\|\mathbf{x}\|^2} \le 1$ generate four polynomial vertices. Quadratic enclosure errors multiply degree-five and degree-four terms, giving contact $\mathbf{F}_\nu - \mathbf{F} = O(\|\mathbf{x}\|^6)$ on the graph, sufficient for Proposition 1.

Three centers: prescribed reference classes. The polynomial analogue of (15) uses $n_{\mathbf{x}} = 3$ and replaces both analytic factors by one. Jet layers have sizes $1, 3, 6$. The coefficients of $\widehat{f}_i$ on x_i^5 have magnitudes in $[3/2, 3]$; its mixed coefficients lie in $[-1/8, 1/8]$. For the leading reduced field g, weighted AM–GM bounds the mixed terms of $\sum_i x_i^3 g_i$ by pure eighth powers. Its strict sign gives 32 stable and eight unstable instances.

Results.

All 34 stable two-center systems and all 32 stable three-center systems receive exact certificates; none of the unstable instances does. Maximum sizes $(\mathbf{Q}_W, \mathbf{R}_{\nu,0}, \mathbf{R}_{\nu,1})$ are $(14, 75, 47)$ and $(20, 95, 42)$, respectively.

The original-coordinate experiment is limited to $n_{\mathbf{y}} = 1$ and uses MATLAB's numerical `null` to implement the prescribed kernel compression.

A physical analytic model. For the spin-1 mean-field Blume–Emery–Griffiths (BEG) model [19] with unit-rate heat-bath relaxation [20], let M_s be magnetization and Q_s the occupied-spin fraction. Set $x = M_s$, $y = Q_s - 1/a$, $a = \beta J > 1$, $b = \beta K$, $\beta\Delta = \ln(2a-2) + b/a$ and $\varrho = 1/(a-1)$. Then

$$\begin{aligned} D_y &= 1 + \varrho e^{by}\cosh(ax), \\ \dot{x} &= -x + \varrho e^{by}\sinh(ax)/D_y, \\ \dot{y} &= -1/a - y + \varrho e^{by}\cosh(ax)/D_y. \end{aligned} \tag{16}$$

On $b = -a(a-3)/(2(a-1))$,

$$T_2 h = \frac{a}{3}x^2, \qquad \bar{f} = \frac{a^4(a^2 - 2a - 9)}{270(a-1)^2}x^5 + O(x^7). \tag{17}$$

Together with the transverse eigenvalue $-3(a-1)/(2a)$, this gives 11 stable and four unstable retained rational parameters; the quintic test does not decide $a = 1 + \sqrt{10}$. We seek finite-neighborhood certificates of the *true field*, using $D = 4$, $\ell = x^8$ and $z = y - ax^2/3$.

Let $\widetilde{\mathbf{F}}$ be this analytic field after the coordinate change and $\mathbf{P} = T_5\widetilde{\mathbf{F}}$. On $x^2 + z^2 \le r^2$, the code certifies

$$\begin{aligned} |\widetilde{F}_i - P_i| \le b_r, \qquad b_r = M(r)(x^2 + z^2)^3, \\ \mathbf{F}_s = \mathbf{P} + b_r \mathbf{s}, \qquad \mathbf{s} \in \{-1, 1\}^2. \end{aligned}$$

It computes degrees six through eleven exactly and bounds the remaining analytic tail by Cauchy estimates on a complex polydisc with a certified nonzero denominator. The four-vertex program then certifies all 11 stable parameters: eight at $r = 1/16$, two at $1/32$, and $a = 4$ at $1/64$. The balls are checked to lie in $|M_s| \le Q_s \le 1$. No unstable instance is certified.

Reproducibility. The repository provides scalar generation, frozen exact inputs for the other families, numerical programs, stored witnesses, rational certificates and exact verifiers. The `witnesses`, `defects`, `verify`, and `tables` actions check the retained evidence and rebuild the tables without an SDP solve.

VI. CONCLUSION

Under the flatness and positive-localizer assumptions, restricted derivative orders force Gram kernels in any invariant-manifold dimension. Rational moments provide explicit exposing certificates, orthogonal to each successive affine coefficient-matching set and computable before solving; adapted coordinates turn the reduction into monomial deletion. Reduced central-point optimization and rational reconstruction yield exact certificates on the tested systems, including true non-polynomial fields.

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